International Journal of Mathematics, Statistics and Operations Research
Vol. 5; Number 2; 2025, pp. 269-285
https://doi.org/10.47509/IJMSOR.2025.v05i02.06

ARF INDIA
Academic Open Access Publishing
www.arfjournals.com

Mixed Quasi Hemiequilibrium Problems on Hadamard Manifolds

Shreyasi Jana[a], Muhammad Aslam Noor[b]

[a] *Department of Mathematics, Narajole Raj College, West Medinipur 721211, West Bengal, India*
[b] *COMSATS University Islamabad, Islamabad, Pakistan*

ABSTRACT
In this article we address a general category of equilibrium problems known as mixed quasi hemiequilibrium problems (MQHEP) on Hadamard manifolds. By using KKM technique, we establish the existence and uniqueness of solutions to MQHEP when the underlying bifunctions are monotone. We provide examples in the Hadamard manifold context to illustrate our results. To tackle mixed quasi hemiequilibrium problems on these nonlinear domains, we additionally look into a few iterative algorithms. Some certain cases of MQHEP are provided. These broad types of equilibrium problems are new on Hadamard manifolds. We hope our results and recommendations will encourage further research in this interesting and fascinating field of study.

KEYWORDS
Riemannian Geometry, Hadamard manifolds, Mixed Quasi Hemiequilibrium problems, KKM mappings.

1. Introduction

We may examine a broad class of problems that arise in finance, economics, network analysis, transportation, elasticity, and optimization using equilibrium problems because they offer a coherent, natural, creative, and general framework. Equilibrium problems and variational inequality problems have been widely applied to study various problems arising in economics, mechanics and engineering science (see, for example [7], [17]). The existence and uniqueness of solutions for equilibrium problems and variational inequality problems have since been the subject of numerous investigations (e.g., [2], [3], [20], [35]).

However, several researchers have been attempting to extend various concepts and methods of nonlinear analysis from Euclidean spaces to Riemannian manifolds (see, for instance, [4], [6], [23], [29], [30], [33]). This is due to the fact that by simply applying the proper Riemannian metric to the underlying manifold, many nonconvex and nonsmooth optimization problems can be regarded as convex and smooth unconstrained optimization problems. The following Example 1.1 is one such example:

CONTACT Author[a]. Email: shreyasi.iitkgp@gmail.com

Article History
Received : 30 September 2025; Revised : 28 October 2025; Accepted : 03 November 2025; Published : 10 November 2025

To cite this paper
Shreyasi Jana & Muhammad Aslam Noor (2025). Mixed Quasi Hemiequilibrium Problems on Hadamard Manifolds. *International Journal of Mathematics, Statistics and Operations Research.* 5(2), 269-285.

Example 1.1. *([6])* Let us consider the Rosenbock's banana function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2;$$

and the vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$V(x) = (-x_1^2 + x_1 + x_2, -2x_1^3 + 2x_1^2 + 2x_1x_2 - x_1),\ x = (x_1, x_2) \in \mathbb{R}^2.$$

This f is not a convex function and V is not a monotone vector field in the classical sense.
Endowing $\mathbb{R}^2$ with the Riemannian metric $G : \mathbb{R}^2 \to S^n_{++}$, such that

$$G(x) = \begin{pmatrix} 4x_1^2 + 1 & -2x_1 \\ -2x_1 & 1 \end{pmatrix},\quad x = (x_1, x_2),$$

we obtain the Riemannian manifold M_G that is complete and of constant curvature 0. Then f becomes a convex function and V transforms into a monotone vector field in M_G.

Colao et al. [5] presented the theory of equilibrium problems on Hadamard manifolds in 2012. Prior research on monotone and accretive vector fields on Riemannian manifolds was done by Németh [22] and Wang et al. [34]. Maximal monotone vector fields were extended from Banach spaces to Hadamard manifolds by Li et al. [18]. Some basic existence and uniqueness theorems of the classical theory of variational inequalities on Hadamard manifolds were extended by Németh [21]. Variational inequalities on Riemannian manifolds were explored by Li et al. [19]. Zhou and Huang [36] established a generalized KKM theorem on Hadamard manifolds and created the concept of KKM mapping. Jana and Nahak ([12], [13]) have studied some existence results for equilibrium and mixed equilibrium problems on these spaces. Other significant and useful generalization of equilibrium problems are hemiequilibrium problems (see for example [26], [27]) and quasi equilibrium problems ([24], [25]). Hemivariational inequality problems on Hadamard manifolds were recently introduced by Tang et al. [32], while Jana and Nahak ([14], [15], [16]) explored the idea of mixed hemivariational inequality problems and hemi equilibrium problems on Hadamard manifolds. As far as we know, no literature has studied mixed quasi hemiequilibrium problems on Hadamard manifolds. This work identifies the conditions under which the solution sets of mixed hemiequilibrium problems are nonempty and contain precisely one point. Additionally, iterative strategies for dealing with mixed quasi hemiequilibrium problems are proposed.
Since the mixed quasi hemieuilibrium problems include the mixed hemiequilibrium problems as special cases, results obtained in this paper continue to hold for these problems. In fact, our results can be viewed as significant extensions and generalizations of the previously known results for several classes of equilibrium problems, variational inequalities and related optimization problems. This work is the initial step to explore mixed quasi hemiequilibrium problems in nonlinear spaces, and the insights it gives will encourage scholars to conduct future research in this fascinating area of mathematics.

2. Preliminaries

In this section, we recollect some basic terminology, fundamental characteristics, and notations required for a comprehensive study of this article. Any textbook on Riemannian geometry would have these (e.g., [31], [33]).

Let M be a connected manifold of dimension n. We designate the tangent bundle of M as $TM = \cup_{x\in M} T_xM$, and the n-dimensional tangent space of M at x as T_xM, respectively. When M has a Riemannian metric $< .,. >$ on the tangent space T_xM and a corresponding norm denoted by $\|.\|$, it is a Riemannian manifold. The length of a piecewise smooth curve $\gamma : [a, b] \to M$ joining x to y such that $\gamma(a) = x$ and $\gamma(b) = y$, is defined by

$$L(\gamma) = \int_a^b \| \dot{\gamma}(t) \|_{\gamma(t)} \, dt.$$

Then for any $x, y \in M$ the original topology on M is persuade by the Riemannian distance $d(x, y)$, which can be determined by the shortest length of all curves connecting x and y.

The Levi-Civita connection, indicated by $\nabla_X Y$ on any Riemannian manifold, is the only covariant derivation for any vector fields X, Y on M. Let γ be a smooth curve in M. A vector field X is said to be parallel along γ if $\nabla_{\gamma'} X = 0$,. If γ' is parallel along γ, then γ is a geodesic. A geodesic is regarded as minimum if its length in M between x and $y = d(x, y)$.

If, for every $x \in M$, all geodesics originating from x are specified for every $t \in \mathbb{R}$, then the Riemannian manifold is complete. Any pair of points in M can be connected by a minimal geodesic if M is complete, according to the Hopf-Rinow theorem. Additionally, bounded closed subsets are compact and (M, d) is a full metric space.The exponential mapping $\exp_x : T_xM \to M$, assuming that M is complete, is defined as $\exp_x v = \gamma_v(1)$, where γ_v is the geodesic characterized by its position x and velocity v at x.

A Hadamard manifold is a simply connected complete Riemannian manifold with non-positive sectional curvature and the exponential mapping exp and its inverse $\exp^{-1}$ are continuous on it.

A geodesic triangle $\Delta(x_1x_2x_3)$ of a Riemannian manifold is the set consisting of three distinct points x_1, x_2, x_3 called the vertices and three minimizing geodesic segments γ_{i+1} joining x_{i+1} to x_{i+2} called the sides, where $i = 1, 2, 3(mod\ 3)$.

Theorem 2.1. *[31] Let M be a Hadamard manifold, $\Delta(x_1x_2x_3)$ a geodesic triangle and $\gamma_{i+1} : [0, l_{i+1}] \to M$ geodesic segments joining x_{i+1} to x_{i+2} and set*

$$l_{i+1} = l(\gamma_{i+1}),\ \theta_{i+1} = \measuredangle(\gamma'_{i+1}(0), -\gamma'_i(l_i)),\ for\ i = 1, 2, 3(mod\ 3).$$

Then

$$\theta_1 + \theta_2 + \theta_3 \le \pi,$$

$$l_{i+1}^2 + l_{i+2}^2 - 2l_{i+1}l_{i+2}\cos\theta_{i+2} \le l_i^2,$$

$$d^2(x_{i+1}, x_{i+2}) + d^2(x_{i+2}, x_i) - 2\big\langle \exp^{-1}_{x_{i+2}} x_{i+1}, \exp^{-1}_{x_{i+2}} x_i \big\rangle \leq d^2(x_i, x_{i+1}). \tag{2.1}$$

By using the above inequality for any three points $x, y, z \in M$, we can get

$$d^2(x, y) \leq \big\langle \exp^{-1}_x z, \exp^{-1}_x y \big\rangle + \big\langle \exp^{-1}_y z, \exp^{-1}_y x \big\rangle.$$

Lemma 2.1.1. *([18]) Let $x_0 \in M$ and $\{x_n\} \in M$ such that $x_n \to x_0$. Then the following assertions hold.*

(i) For any $y \in M$

$$\exp^{-1}_{x_n} y \to \exp^{-1}_{x_0} y \text{ and } \exp^{-1}_y x_n \to \exp^{-1}_y x_0.$$

(ii) If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \to v_0$, then $v_0 \in T_{x_0}M$.
(iii) Given the sequence $\{u_n\}$ and $\{v_n\}$ with $u_n, v_n \in T_{x_n}M$, if $u_n \to u_0$ and $v_n \to v_0$ with $u_0, v_0 \in T_{x_0}M$, then $\langle u_n, v_n \rangle \to \langle u_0, v_0 \rangle$.

Definition 2.2. *([30])* Let M represent a Hadamard manifold with finite dimensions. A subset K of M is said to be geodesic convex if and only if for any two points $x, y \in K$, the geodesic joining x to y is contained in K. That is if $\gamma : [0, 1] \to M$ is a geodesic with $x = \gamma(0)$ and $y = \gamma(1)$, then $\gamma(t) \in K$, *for* $0 \leq t \leq 1$.

Definition 2.3. *([30])* A real-valued function $f : M \to \mathbb{R}$ defined on a geodesic convex set K is said to be geodesic convex if and only if for $0 \leq t \leq 1$,

$$f(\gamma(t)) \leq (1 - t)f(\gamma(0)) + tf(\gamma(1)).$$

Definition 2.4. *([5])* For an arbitrary subset $C \subseteq M$ the minimal geodesic convex subset which contains C is called the convex hull of C and is denoted by $co(C)$. It is easy to check that

$$co(C) = \bigcup_{n=1}^{\infty} C_n, \text{ where } C_0 = C \text{ and } C_n = \{z \in \gamma_{x,y} : x, y \in C_{n-1}\}.$$

Definition 2.5. *([37])* Let M be a finite dimensional Hadamard manifold. Let $K \subset M$ be a nonempty closed geodesic convex set and $G : K \to 2^K$ be a set-valued mapping. We say that G is a KKM mapping if for any $\{u_1, ..., u_m\} \subset K$, we have

$$co(\{u_1, ..., u_m\}) \subset \bigcup_{i=1}^{m} G(u_i).$$

Lemma 2.5.1. *([5]) Let K be a nonempty closed geodesic convex set and $G : K \to 2^K$ be a set-valued mapping such that for each $u \in K$, $G(u)$ is closed. Suppose that*

(i) there exists $u_0 \in K$ such that $G(u_0)$ is compact.
(ii) $\forall u_1, .., u_m \in K$, $co(\{u_1, .., u_m\}) \subset \bigcup_{i=1}^{m} G(u_i)$.

Then $\bigcap_{u \in K} G(u) \neq \emptyset$.

Definition 2.6. *([1], [10], [28])* Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a proper function. It is said to be a locally Lipschitz function on M if for each $x \in dom f$, there exist ϵ_x and

$L_x > 0$ such that

$$|f(z) - f(y)| \leq L_x d(z, y), \ \forall z, y \in B(x, \epsilon_x),$$

where $B(x, \epsilon_x)$ denotes an open ball centered in $x \in M$ and radius ϵ_x.

Definition 2.7. *([10])* Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitz function on M. Given $x \in domf$, the generalized directional derivative in the sense Clarke of f at the point p in the direction $w \in T_pM$, denoted by $f^o(p; w)$, is defined as

$$f^o(p; w) = \lim_{t \to 0^+} \sup_{q \to p} \frac{fo\phi^{-1}(\phi(q) + td\phi(p)w) - fo\phi^{-1}(\phi(q))}{t},$$

where (ϕ, U) is a chart at p.

We require the following lemma which provides some fundamental characteristics of the generalized directional derivative on Hadamard manifolds.

Lemma 2.7.1. *([10]) Let M be a Riemannian manifold and $p \in M$. Suppose that the function $f : M \to R$ is Lipschitz of rank K on an open neighborhood U of p. Then,*

(i) for each $q \in U$, the function $w \to f^o(q; w)$ is finite, positive homogeneous and subadditive on T_qM, and satisfies

$$|f^o(q; w)| \leq K\|w\|;$$

(ii) $f^o(q; w)$ is upper semicontinuous on TM and as a function of w alone is Lipschitz of rank K on T_qM for each $q \in U$;

(iii) $f^o(q; -w) = (-f)^o(q; w)$ for each $q \in U$ and $w \in T_qM$.

In the rest of the work, unless otherwise specified, we assume that M is a finite dimensions Hadamard manifold and that $K \subseteq M$ is a nonempty closed geodesic convex set.

3. Main Results

The existence and uniqueness of solutions to mixed quasi hemiequilibrium problems (MQHEP) on Hadamard manifolds are determined in this section.
Let M be a Hadamard manifold and K a closed geodesic convex subset of M. Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying the property $F(u, u) = 0$, for all $u \in K$. Then the equilibrium problem introduced by Colao et al. [5] is to find a point $u \in K$, such that

$$(EP) \quad F(u, v) \geq 0, \ \forall v \in K. \tag{3.1}$$

We introduce mixed quasi hemiequilibrium problem on Hadamard manifolds.
Suppose that K is a closed geodesic convex subset of the Hadamard manifold M. Let $F : K \times K \to \mathbb{R}$ be a bifunction such that for every $u \in K$, $F(u, u) = 0$,. Assume that $\psi(.,.) : K \times K \to \mathbb{R}$ is a bifunction and $J : M \to \mathbb{R}$ is a locally Lipschitz function. Then the mixed quasi hemiequilibrium problem, denoted by MQHEP(F,J,K) is to

find an element $u \in K$ such that

$$F(u,v) + J^o(u; \exp_u^{-1} v) + \psi(v,u) - \psi(u,u) \geq 0, \ \forall v \in K. \tag{3.2}$$

We denote the solution set of MQHEP(F,J,K) by SOL(MQHEP).

Definition 3.1. ([5]) We call a bifunction F to be monotone on K, if for any $u, v \in K$, we have

$$F(u,v) + F(v,u) \leq 0. \tag{3.3}$$

Definition 3.2. ([27]) $J^o(;)$ is said to be monotone if

$$J^o(u; \exp_u^{-1} v) + J^o(v; \exp_v^{-1} u) \leq 0. \tag{3.4}$$

Definition 3.3. ([25], [11]) The bifunction $\psi(.,.)$ is said to be skew-symmetric if for any $u, v \in K$, we have

$$\psi(u,u) - \psi(u,v) - \psi(v,u) + \psi(v,v) \geq 0, \forall u, v \in K. \tag{3.5}$$

Definition 3.4. Let K be a geodesic convex subset of M. A function $f : K \to \mathbb{R}$ is said to be hemicontinuous if for every geodesic $\gamma : [0,1] \to K$, whenever $t \to 0$, $f(\gamma(t)) \to f(\gamma(0))$.

3.1. *Existence Results for Mixed Quasi Hemiequilibrium Problems*

Lemma 3.4.1. *Assume that $F : K \times K \to \mathbb{R}$ be monotone and hemicontinuous in the first argument. Let for fixed $u \in K$, the mapping $z \mapsto F(u,z)$ be geodesic convex and $J : M \to \mathbb{R}$ a locally Lipschitz function. Suppose $\psi(.,.) : K \times K \to \mathbb{R}$ is geodesic convex in first argument. Then $u \in K$ satisfies*

$$F(u,v) + J^o(u; \exp_u^{-1} v) + \psi(v,u) - \psi(u,u) \geq 0, \ \forall v \in K; \tag{3.6}$$

if and only if

$$-F(v,u) + J^o(u; \exp_u^{-1} v) + \psi(v,u) - \psi(u,u) \geq 0, \ \forall v \in K. \tag{3.7}$$

Proof. First we show (3.6) implies (3.7). Since F is monotone

$$F(u,v) + F(v,u) \leq 0,$$

or,

$$F(u,v) \leq -F(v,u).$$

i.e.,

$$F(u,v)+J^o(u; \exp_u^{-1} v)+\psi(v,u)-\psi(u,u) \leq -F(v,u)+J^o(u; \exp_u^{-1} v)+\psi(v,u)-\psi(u,u).$$

By (3.6)

$$-F(v,u) + J^o(u; \exp_u^{-1} v) + \psi(v,u) - \psi(u,u) \geq 0.$$

Therefore (3.6) implies (3.7).
Next to show (3.7) implies (3.6). Let $u \in K$ be a solution of (3.7). Then

$$-F(v,u) + J^o(u; \exp_u^{-1} v) + \psi(v,u) - \psi(u,u) \geq 0, \ \forall v \in K. \tag{3.8}$$

Let $w \in K$ be arbitrarily fixed and $\gamma(t) = \exp_u(t \exp_u^{-1} w)$ for $t \in [0,1]$ be a geodesic joining u and w.
As K is geodesic convex, then $\gamma(t) \in K$, for $t \in [0,1]$. It follows from (3.8)

$$-F(\gamma(t),u) + J^o(u; \exp_u^{-1} \gamma(t)) + \psi(\gamma(t),u) - \psi(u,u) \geq 0, \ for \ 0 \leq t \leq 1. \tag{3.9}$$

As $\psi(.,.)$ is geodesic convex in the first argument, then

$$\psi(\gamma(t),u) \leq t\psi(w,u) + (1-t)\psi(u,u)$$

$$or, \ \psi(\gamma(t),u) - \psi(u,u) \leq t[\psi(w,u) - \psi(u,u)].$$

Now

$$0 = F(\gamma(t),\gamma(t)) \leq tF(\gamma(t),w) + (1-t)F(\gamma(t),u), \ (as \ z \mapsto F(u,z) \ is \ geodesic \ convex,)$$

$$or, \ t[F(\gamma(t),u) - F(\gamma(t),w)] \leq F(\gamma(t),u)),$$

$$or, \ t[F(\gamma(t),u) - F(\gamma(t),w)] \leq J^o(u; \exp_u^{-1} \gamma(t)) + \psi(\gamma(t),u) - \psi(u,u), \ by \ (3.9).$$

As $\psi(.,.)$ is geodesic convex in the first argument,

$$t[F(\gamma(t),u) - F(\gamma(t),w)] \leq J^o(u; \exp_u^{-1} \exp_u(t \exp_u^{-1} w)) + t[\psi(w,u) - \psi(u,u)].$$

Also by the positively homogeneous property of $J^o(u; t \exp_u^{-1} w)$ [see Lemma 2.7.1], we have

$$t[F(\gamma(t),u) - F(\gamma(t),w)] \leq tJ^o(u; \exp_u^{-1} w) + t[\psi(w,u) - \psi(u,u)],$$

which implies,

$$F(\gamma(t),u) - F(\gamma(t),w) \leq J^o(u; \exp_u^{-1} w) + \psi(w,u) - \psi(u,u), \ (as \ t > 0).$$

Since F is hemicontinuous in the first argument taking $t \to 0$, we have

$$F(u,u) - F(u,w) \leq J^o(u; \exp_u^{-1} w) + \psi(w,u) - \psi(u,u), \ \forall w \in K;$$

$$or,\ F(u,w) + J^o(u; \exp_u^{-1} w) + \psi(w,u) - \psi(u,u) \geq 0,\ \forall\ w \in K.$$

This completes the proof. □

Next, we prove the primary existence theorem. First, we take the set K to be bounded, so in this case, K is compact.

Theorem 3.5. *Let K be a compact subset of M. Also let $F : K \times K \to \mathbb{R}$ be monotone and hemicontinuous in the first argument. Suppose for fixed $u \in K$, the mapping $z \mapsto F(u,z)$ is geodesic convex, lower semicontinuous and $J : M \to \mathbb{R}$ is a locally Lipschitz function. Also assume that $\psi(.,.) : K \times K \to \mathbb{R}$ is skew symmetric, geodesic convex in the first argument and lower semicontinuous. Then MQHEP(F,J,K) admits unique solution.*

Proof. Uniqueness of solution:
If possible let the problem has two different solutions say u_1 and u_2. Then

$$F(u_1,v) + J^o(u_1; \exp_{u_1}^{-1} v) + \psi(v,u_1) - \psi(u_1,u_1) \geq 0, \forall v \in K. \tag{3.10}$$

$$F(u_2,v) + J^o(u_2; \exp_{u_2}^{-1} v) + \psi(v,u_2) - \psi(u_2,u_2) \geq 0, \forall v \in K. \tag{3.11}$$

Now putting $v = u_2$ in (3.10) and $v = u_1$ in (3.11) and adding we get
$F(u_1,u_2) + J^o(u_1; \exp_{u_1}^{-1} u_2) + \psi(u_2,u_1) - \psi(u_1,u_1) + F(u_2,u_1) + J^o(u_2; \exp_{u_2}^{-1} u_1)$
$+\psi(u_1,u_2) - \psi(u_2,u_2) \geq 0,$
$or, F(u_1,u_2) + F(u_2,u_1) + J^o(u_1; \exp_{u_1}^{-1} u_2) + J^o(u_2; \exp_{u_2}^{-1} u_1)$
$\geq \psi(u_1,u_1) - \psi(u_2,u_1) - \psi(u_1,u_2) + \psi(u_2,u_2) \geq 0$, (by skew symmetric property of ψ).
But since F and J are monotone we have

$$F(u_1,u_2) + F(u_2,u_1) + J^o(u_1; \exp_{u_1}^{-1} u_2) + J^o(u_2; \exp_{u_2}^{-1} u_1) \leq 0.$$

Hence we have

$$F(u_1,u_2) + F(u_2,u_1) + J^o(u_1; \exp_{u_1}^{-1} u_2) + J^o(u_2; \exp_{u_2}^{-1} u_1) = 0.$$

Therefore $u_1 = u_2$.
This completes the proof.
Existence of solution:
Consider the two set-valued mappings $G_1 : K \to 2^K$ and $G_2 : K \to 2^K$ such that

$$G_1(v) = \{u \in K : F(u,v) + J^o(u; \exp_u^{-1} v) + \psi(v,u) - \psi(u,u) \geq 0\},\ \forall\ v \in K.$$

$$G_2(v) = \{u \in K : -F(v,u) + J^o(u; \exp_u^{-1} v) + \psi(v,u) - \psi(u,u) \geq 0\},\ \ \forall\ v \in K.$$

It is easy to see that $u \in K$ solves MQHEP(F,J,K) if and only if $u \in \cap_{v \in K} G_1(v)$. Thus it suffices to prove that $\cap_{v \in K} G_1(v) \neq \emptyset$.
Step-1: G_1 is a KKM map.

So we have to prove that for any choice of $v_1, v_2, ..., v_m \in K$,

$$co(\{v_1, ..., v_m\}) \subset \bigcup_{i=1}^{m} G_1(v_i).$$

Suppose on the contrary that there exists a point u_0 *in* K, such that $u_0 \in co(\{v_1, ..., v_m\})$ but $u_0 \notin \bigcup_{i=1}^{m} G_1(v_i)$. That is

$$F(u_0, v_i) + J^o(u_0; \exp_{u_0}^{-1} v_i) + \psi(v_i, u_0) - \psi(u_0, u_0) < 0, \ \forall i \in \{1, ..., m\}.$$

This implies that for any

$$i \in \{1, ..., m\}, \ v_i \in \{v \in K : F(u_0, v) + J^o(u_0; \exp_{u_0}^{-1} v) + \psi(v, u_0) - \psi(u_0, u_0) < 0\}.$$

Since the function $v \mapsto F(u_0, v)$ and $v \mapsto \psi(v, u_0)$ are geodesic convex. Being the sum of two convex function $v \mapsto F(u_0, v) + \psi(v, u_0)$ is geodesic convex. Therefore, the set $\{v \in K : F(u_0, v) + J^o(u_0; \exp_{u_0}^{-1} v) + \psi(v, u_0) - \psi(u_0, u_0) < 0\}$ is a geodesic convex set. Then

$$u_0 \in co(\{v_1, ..., v_m\}) \subseteq \{v \in K : F(u_0, v) + J^o(u_0; \exp_{u_0}^{-1} v) + \psi(v, u_0) - \psi(u_0, u_0) < 0\}.$$

Therefore,

$$F(u_0, u_0) + J^o(u_0; \exp_{u_0}^{-1} u_0) + \psi(u_0, u_0) - \psi(u_0, u_0) < 0.$$

But we have,

$$F(u_0, u_0) + J^o(u_0; \exp_{u_0}^{-1} u_0) = 0,$$

a contradiction. Hence G_1 is a KKM mapping.
Step-2: To show G_2 is a KKM map.
From Lemma 3.4.1, we have $G_1(v) \subset G_2(v), \ \forall v \in K$. That is,

$$co(\{v_1, v_2, ..., v_m\}) \subset \bigcup_{i=1}^{m} G_2(v_i).$$

Hence G_2 is also a KKM mapping.
Step-3: Next we show $G_2(v)$ is closed.
Let $\{u_n\} \in G_2(v)$, such that $u_n \to u$ as $n \to \infty$. We show that $u \in G_2(v)$, Since $\{u_n\} \in G_2(v)$, we have

$$-F(v, u_n) + J^o(u_n; \exp_{u_n}^{-1} v) + \psi(v, u_n) - \psi(u_n, u_n) \geq 0.$$

Since $F(v, .)$ is lower semicontinuous, $\psi(., .)$ is lower semicontinuous and J^o is Lipschitz continuous, we have

$$-F(v, u) + J^o(u; \exp_u^{-1} v) + \psi(v, u) - \psi(u, u) \geq 0.$$

Hence $u \in G_2(v)$. That is $G_2(v)$ is closed for all $v \in K$.
Step-4: $G_2(v)$ is compact.
Since $G_2(v)$ is a closed subset of a compact set K. So $G_2(v)$ is compact for all $v \in K$.
Hence by Lemma 2.5.1, there exists a point $u \in K$ such that $u \in \bigcap_{v \in K} G_2(v)$.
By Lemma 3.4.1, we have

$$\bigcap_{v \in K} G_1(v) = \bigcap_{v \in K} G_2(v).$$

That is $u \in \bigcap_{v \in K} G_1(v)$.
So there exists a point $u \in K$ such that

$$F(u,v) + J^o(u; \exp_u^{-1} v) + \psi(v,u) - \psi(u,u) \geq 0, \ \forall v \in K.$$

Therefore, $u \in K$ solves MQHEP(F,J,K). □

Example 3.6. Let K be a subset of $\mathbb{R}^2$ defined by

$$K = \{u = (u_1, u_2) \in \mathbb{R}^2_+ : u_1^2 + u_2^2 \leq 4 \leq (u_1 - 1)^2 + u_2^2\}.$$

K is not convex in $\mathbb{R}^2$ (see Example 3.1 of [32]). We consider the Poincare upper half model

$$H^2 = \{u = (u_1, u_2) \in \mathbb{R}^2; u_2 > 0\};$$

which forms a Hadamard manifold with constant curvature -1. Now the set $K \subset H^2$ is geodesic convex and compact with respect to the metric defined by

$$g_{H^2} = \frac{\delta_{ij}}{u_2^2}.$$

Now we define the bifunction $F : K \times K \to \mathbb{R}$ by

$$F(u,v) = v_2 - u_2.$$

Therefore, $F(u,v) + F(v,u) = v_2 - u_2 + u_2 - v_2 = 0$.
Hence F is monotone on K. Let $\psi : K \times K \to \mathbb{R}$ be a constant map. It is clear that $z \mapsto F(u,z)$ is geodesic convex, lower semicontinuous.
Also $\psi : K \times K \to \mathbb{R}$ is skew-symmetric, geodesic convex, lower semicontinuous.
Let $J : H^2 \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$J(u) = \ln u_2.$$

Then J is locally Lipschitz on H^2 (see Example 5 of [8]).
Then by Theorem 3.5, the MQHEP(F,J,K) has unique solution.

Next we consider K to be unbounded. So in this case K is noncompact.

Theorem 3.7. *Let $F : K \times K \to \mathbb{R}$ be monotone and hemicontinuous in the first argument. Suppose for fixed $u \in K$, the mapping $z \mapsto F(u,z)$ is geodesic convex, lower semicontinuous and $J : M \to \mathbb{R}$ is a locally Lipschitz function. Also assume that*

$\psi(.,.) : K \times K \to \mathbb{R}$ is geodesic convex in the first argument, lower semicontinuous and skew-symmetric. If there exists a point $v_0 \in K$, such that

$$F(u, v_0) + J^o(u; \exp_u^{-1} v_0) + \psi(v_0, u) - \psi(u, u) < 0, \text{ whenever } d(\mathbf{0}, u) \to +\infty, \ u \in K, \tag{3.12}$$

then MQHEP(F,J,K) has unique solution.

Proof. Uniqueness of solution is same as Theorem 3.5. Here we prove the existence of solution.
Given a point $\mathbf{0} \in M$, we denote $\Sigma_R = \{u \in M : d(\mathbf{0}, u) \leq R\}$ to be the closed geodesic ball of radius R and center $\mathbf{0}$. Let $K_R = K \cap \Sigma_R$. If $K_R \neq \emptyset$, then there exists at least one $u_R \in K_R$ such that

$$F(u_R, v) + J^o(u_R; \exp_{u_R}^{-1} v) + \psi(v, u_R) - \psi(u_R, u_R) \geq 0, \ \forall v \in K_R, \tag{3.13}$$

by Theorem 3.5.
We now take a point $v_0 \in K$ satisfying (3.12), with $d(\mathbf{0}, v_0) < R$, so $v_0 \in K_R$.
Hence by (3.13), we have

$$F(u_R, v_0) + J^o(u_R; \exp_{u_R}^{-1} v_0) + \psi(v_0, u_R) - \psi(u_R, u_R) \geq 0. \tag{3.14}$$

If $d(\mathbf{0}, v_R) = R$ for all R, we may choose R large enough so that $d(\mathbf{0}, v_R) \to +\infty$.
Hence by (3.12),

$$F(u_R, v_0) + J^o(u_R; \exp_{u_R}^{-1} v_0) + \psi(v_0, u_R) - \psi(u_R, u_R) < 0,$$

contradicts (3.14).
So there exists an R such that $d(\mathbf{0}, v_R) < R$.
Given $v \in K$, let $\gamma(t) = \exp_{u_R}(t \exp_{u_R}^{-1} v)$ be a geodesic joining u_R to v. Now since $d(\mathbf{0}, u_R) < R$, we can choose $0 < t < 1$, sufficiently small so that $\gamma(t) \in K_R$.
Hence

$$0 \leq F(u_R, \gamma(t)) + J^o(u_R; \exp_{u_R}^{-1} \gamma(t)) + \psi(\gamma(t), u_R) - \psi(u_R, u_R),$$

$$i.e., 0 \leq tF(u_R, v) + (1-t)F(u_R, u_R) + J^o(u_R; t \exp_{u_R}^{-1} v) + t(\psi(v, u_R) - \psi(u_R, u_R)),$$

$$= t[F(u_R, v) + J^o(u_R; \exp_{u_R}^{-1} v) + \psi(v, u_R) - \psi(u_R, u_R)], \ [by\ Lemma\ 2.7.1]$$

or, as $t > 0$,

$$F(u_R, v) + J^o(u_R; \exp_{u_R}^{-1} v) + \psi(v, u_R) - \psi(u_R, u_R) \geq 0, \ for\ v \in K.$$

That is u_R solves MQHEP(F,J,K). □

3.2. *Proximal Point Algorithm for Solving Mixed Quasi Hemiequilibrium Problem*

Jana and Nahak [13] have studied certain methods for handling mixed equilibrium problems on these spaces. We shall now discuss the proximal point algorithm (PPA) for the mixed quasi hemiequilibrium problem (3.2).
We assume K to be compact geodesic convex subset of the Hadamard manifold M.
At stage n, given $u_n \in K$, $\rho > 0$, compute $u_{n+1} \in K$, as a solution of the mixed quasi hemiequilibrium problem

$$F(u_{n+1}, v)+\frac{1}{\rho}\langle \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} v\rangle+J^o(u_{n+1}; \exp_{u_{n+1}}^{-1} v)+\psi(v, u_{n+1})-\psi(u_{n+1}, u_{n+1}) \geq 0,\ \forall v \in K. \tag{3.15}$$

The principle of Fejér convergence and the associated findings, which are available in [9] and [18], are then revisited.

Definition 3.8. Let X be a complete metric space and $A \subseteq X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is said to be Fejér convergent to A if

$$d(x_{n+1}, y) \leq d(x_n, y),\ \forall y \in A \textit{ and } n = 0, 1, 2, ...$$

Lemma 3.8.1. *Let X be a complete metric space and let A be a nonempty subset of X. Suppose $\{x_n\} \subset X$ be Fejér convergent to K and any cluster point of $\{x_n\}$ belongs to A. Then $\{x_n\}$ converges to a point of A.*

We are now in a position to demonstrate the convergence of PPA for mixed quasi hemiequilibrium problems involving monotone vector fields.

Theorem 3.9. *Let F be monotone and continuous in the first argument and SOL(MQHEP) $\neq \emptyset$. Also assume that the sequence $\{u_n\}$ generated by* (3.15) *is well defined, $\psi(.,.)$ is continuous and skew symmetric. Also assume that $J^o(;)$ is monotone. Then $\{u_n\}$ converges to a solution of the mixed quasi hemiequilibrium problem* (3.2)*.*

Proof. We first proof that $\{u_n\}$ is Fejér convergent to SOL(MQHEP). Let $v \in K$ be a solution of (3.2). Then

$$F(u, v) + J^o(u; \exp_u^{-1} v) + \psi(v, u) - \psi(u, u) \geq 0, \forall v \in K. \tag{3.16}$$

Taking $v = u_{n+1}$ in (3.16), we get

$$F(u, u_{n+1}) + J^o(u; \exp_u^{-1} u_{n+1}) + \psi(u_{n+1}, u) - \psi(u, u) \geq 0. \tag{3.17}$$

Since F is monotone then

$$F(u_{n+1}, u) \leq -F(u, u_{n+1}).$$

Again since ψ is skew-symetric we have

$$\psi(u, u) - \psi(u, u_{n+1}) - \psi(u_{n+1}, u) + \psi(u_{n+1}, u_{n+1}) \geq 0;$$

$$or, \psi(u,u) - \psi(u,u_{n+1}) \geq \psi(u_{n+1},u) - \psi(u_{n+1},u_{n+1}).$$

From (3.15), taking $v = u$ we have

$$F(u_{n+1},u)+\frac{1}{\rho}\big\langle \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} u\big\rangle+J^o(u_{n+1};\exp_{u_{n+1}}^{-1} u)+\psi(u,u_{n+1})-\psi(u_{n+1},u_{n+1}) \geq 0;$$

$$or,\ \frac{1}{\rho}\big\langle \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} u\big\rangle \geq -[F(u_{n+1},u)+J^o(u_{n+1};\exp_{u_{n+1}}^{-1} u)+\psi(u,u_{n+1})-\psi(u_{n+1},u_{n+1})];$$

$$or,\ \frac{1}{\rho}\big\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} u\big\rangle \leq F(u_{n+1},u)+J^o(u_{n+1};\exp_{u_{n+1}}^{-1} u)+\psi(u,u_{n+1})-\psi(u_{n+1},u_{n+1}),$$

$$\leq -F(u,u_{n+1})-J^o(u;\exp_{u}^{-1} u_{n+1})+\psi(u,u_{n+1})-\psi(u_{n+1},u_{n+1})$$

(by monotonicity),

$$or,\ \frac{1}{\rho}\big\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} u\big\rangle \leq -F(u,u_{n+1})-J^o(u;\exp_{u}^{-1} u_{n+1})+\psi(u,u)-\psi(u_{n+1},u),$$

(by skew-symmetric property of ψ),

$$\leq 0,\ by\ (3.17).$$

So we finally get as $\rho > 0$,

$$\big\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} x\big\rangle \leq 0. \tag{3.18}$$

Considering the geodesic triangle $\Delta(u_n u_{n+1} u)$ from (2.1), we get

$$d^2(u_{n+1},u) + d^2(u_{n+1},u_n) - 2\big\langle \exp_{u_{n+1}}^{-1} u_n, \exp_{u_{n+1}}^{-1} u\big\rangle \leq d^2(u_n,u).$$

It follows from (3.18)

$$d^2(u_{n+1},u) + d^2(u_{n+1},u_n) \leq d^2(u_n,u). \tag{3.19}$$

This clearly implies that $d^2(u_{n+1},u) \leq d^2(u_n,u)$, so $\{u_n\}$ is Fejér convergent to SOL(MQHEP). From (3.19) it follows that

$$d^2(u_{n+1},u_n) \leq d^2(u_n,u) - d^2(u_{n+1},u). \tag{3.20}$$

Since the sequence $\{d(u_n,u)\}$ is bounded and monotone, it is also convergent. Hence by (3.20),

$$\lim_{n\to\infty} d^2(u_{n+1},u_n) = 0.$$

That is,

$$\lim_{n\to\infty} d(u_{n+1}, u_n) = 0.$$

Next we prove that any cluster point of $\{u_n\}$ belongs to SOL(MQHEP). Let u be a cluster point of $\{u_n\}$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $u_{n_k} \to u$. Hence $d(u_{n_k+1}, u_{n_k}) \to 0$, by the assertion just proved, and so $u_{n_k+1} \to u$. It follows from (3.15) with $n = n_k$,
$F(u_{n_k+1}, v) + \frac{1}{\rho}\langle \exp^{-1}_{u_{n_k}} u_{n_k+1}, \exp^{-1}_{u_{n_k+1}} v\rangle + J^o(u_{n_k+1}; \exp^{-1}_{u_{n_k+1}} v)$
$+\psi(v, u_{n_k+1}) - \psi(u_{n_k+1}, u_{n_k+1}) \geq 0, \forall v \in K.$
Passing to the limit as $k \to \infty$ in the above equation we get

$$F(u, v) + J^o(u; \exp^{-1}_u v) + \psi(v, u) - \psi(u, u) \geq 0, \forall v \in K.$$

That is $u \in SOL(MQHEP)$. Hence by Lemma 3.8.1, $\{u_n\}$ converges to point of SOL(MQHEP). This completes the proof. □

4. Applications

Mixed quasi hemiequilibrium problems include mixed hemiequilibrium problems, hemiequilibrium problems, hemivariational inequality problems, variational inequality problems, equilibrium problems and optimization problems as particular cases.

(i) Mixed Hemiequilibrium problem: If we define $\psi(v, u) \equiv \psi(v)$, then MQHEP(F,J,K) reduces to the mixed hemiequilibrium problem ([16]), which is to find a point $u \in K$ such that

$$F(u, v) + J^o(u; \exp^{-1}_u v) + \psi(v) - \psi(u) \geq 0, \ \forall v \in K.$$

(ii) Hemiequilibrium problem: If we define $\psi \equiv 0$, then MQHEP(F,J,K) reduces to the hemiequilibrium problem ([15]), which is to find a point $u \in K$ such that

$$F(u, v) + J^o(u; \exp^{-1}_u v) \geq 0, \ \forall v \in K.$$

(iii) Hemivariational inequality problem: For each $u \in K$, let $A : K \to TM$ be a vector field, that is, $A(u) \in T_uM$. Tang et al. ([32]), proposed hemivariational inequality problem on K, which is to find a point $u \in K$ such that

$$\langle A(u), \exp^{-1}_u v\rangle + J^o(u; \exp^{-1}_u v) \geq 0, \ \forall v \in K. \tag{4.1}$$

If we define

$$F(u, v) = \langle A(u), \exp^{-1}_u v\rangle$$

and $\psi \equiv 0$, then MQHEP(F,J,K) and the hemivariational inequality problem (4.1) are equivalent.

(iii) Equilibrium problem: If the function J is constant, then $J^o(u; .) = 0 \in T_uM$ and $\psi \equiv 0$. Then MQHEP(F,J,K) reduces to the following equilibrium problem introduced by Colao et al. ([5]), which is to find $u \in K$ such that

$$F(u,v) \geq 0, \ for \ all \ v \in K,$$

(iv) Variational inequality problem: Assume that $V : K \to TM$ is a vector field, that is, $V_u \in T_uM$ for each $u \in K$. Then the problem introduced by Németh ([21]), is to find $u \in K$ such that

$$\left\langle V_u, \exp_u^{-1} v \right\rangle \geq 0, \ \forall v \in K, \tag{4.2}$$

is called a variational inequality problem on K. Let us denote

$$F(x,y) = \left\langle V_u, \exp_u^{-1} v \right\rangle.$$

If the function J is constant and $\psi \equiv 0$, then MQHEP(F,J,K) and the variational inequality problem (4.2) are same.

(v) If M is a linear space and $\psi \equiv 0$, MQHEP(F,J,K) reduces to the problem, find a point $u \in K$ such that

$$F(u,v) + J^o(u; v-u) \geq 0, \ \forall v \in K,$$

which is a hemiequilibrium problem on Banach spaces introduced and investigated by Noor ([26], [27]).

(vi) Optimization problem: Let $f : K \to \mathbb{R}$ be a function and consider the minimization problem

$$(P) \quad find \ x \in K \ such \ that \ f(x) = \min_{y \in K} f(y).$$

If we set

$$F(x,y) = f(y) - f(x), \ \forall x, y \in K;$$

then the problems (P) and equilibrium problems are identical.

5. Concluding Remarks

To the best of our knowledge, mixed quasi hemiequilibrium problems on Hadamard manifolds have never been introduced before. Future research has a lot of potential, for instance:

(i) From theoretical point of view, one can investigate existence results by applying weaker monotonicity assumptions on the underlying bifunctions.
(ii) Different algorithms can be searched for solving mixed quasi hemiequilibrium problems.
(iii) We have explored the results on Hadamard manifolds. One can try to extend these findings on Riemannian manifolds.
(iv) One can develop and analyze other general classes of equilibrium problems on these nonlinear spaces.

This article can be considered as a stepping stone to look over mixed quasi hemiequilibrium problems on Hadamard Manifolds. We anticipate further fruitful research in this area in near future.

References

[1] Bento, G.C., Ferreira, O.P., Oliveira, P.R., 2010. Local convergence of the proximal point method for a special class of nonconvex functions on Hadamard manifolds. Nonlinear Appl. 73, 564-572.

[2] Bianchi, M. Schaible, S., 1996. Generalized monotone bifunctions and equilibrium problems. J. Optim. Theory Appl. 90, 31-43.

[3] Blum, E., Oettli, W., 1994. From optimization and variational inequalities to equilibrium problems. Math. Student. 63, 123-145.

[4] Bridson, M., Haefliger, A., 1999. Metric spaces of non-positive curvature. Springer-Verlag, Berlin, Heidelberg, New York.

[5] Colao, V., López, G., Marino, G., Martín-Márquez, V., 2012. Equilibrium problems in Hadamard manifolds. J. Math. Anal. Appl. 388, 61-77.

[6] Cruz Neto, J.X.D., Ferreira, O.P., Pérez, L.R.L., Németh, S.Z., 2006. Convex- and monotone-transformable mathamatical programming problems and a proximal-like point method. J. Glob. Optim. 35, 53-69.

[7] Facchinei, F., Pang, J.S., 2003. Finite-dimensional variational inequalities and complementary problems. New York (NY), Springer-Verlag.

[8] Ferreira, O.P., 2008. Dini derivative and a characterization for Lipschitz and convex functions on Riemannian manifolds. Nonlinear Anal. 68, 1517-1528.

[9] Ferreira, O.P., Oliveira, P.R., 2002. Proximal point algorithm on Riemannian manifolds. Optimization 51, 257-270.

[10] Hosseini, S., Pouryayevali, M.R., 2011. Generalized gradients and characterization of epi-Lipschitz sets in Riemannian manifolds. Nonlinear Anal. 74, 3884-3895.

[11] Hung, N.V., Tam, V.M., Pitea, A., 2020. Global error bounds for mixed Quasi-Hemivariational inequality problems on Hadamard manifolds. Optimization doi: DOI:10.1080/02331934.2020.1718126.

[12] Jana, S. Nahak, C., 2016. Mixed equilibrium problems on Hadamard manifolds. Rend. Circ. Mat. Palermo(2). 65(1), 97-109.

[13] Jana, S. Nahak, C., 2017. Equilibrium and Mixed Equilibrium problems on Hadamard Manifolds. Int. J. Mathematics in Operational Research 11(4), 480-496.

[14] Jana, S. Nahak, C., 2021. An introduction to mixed hemivariational inequality problems on Hadamard manifolds. SeMA Journal 78(4), 557-569.

[15] Jana, S. Nahak, C., 2025. Hemi Equilibrium Problems on Hadamard Manifold. Electron. J. Math. Anal. Appl 13(1), 1-12.

[16] Jana, S., 2025. On mixed hemiequilibrium problems in Hadamard manifolds. J. Adv. Math. Stud. 18(2), 190-201.

[17] Kassay, G., Radulescu, V.D., 2015. Equilibrium Problems and Applications. URL: `https://doi.org/10.1016/C2015-0-06685-0`.

[18] Li, C., López, G., Martín-Márquez, V., 2009. Monotone vector fields and the proximal point algorithm on Hadamard manifolds. J. Lond. Math. Soc. 79(2), 663-683.

[19] Li, S.L.,Li, C., Liou, Y.C., Yao, J.C., 2009. Existence of solutions for variational inequalities on Riemannian manifolds. Nonlinear Anal. 71(11), 5695-5706.

[20] Luc, D.T., 2001. Existence results for densely pseudomonotone variational inequalities. J. Math. Anal. Appl. 254, 309-320.

[21] Németh, S.Z., 2003. Variational inequalities on Hadamard manifolds. Nonlinear Anal. 52(5), 1491-1498.

[22] Németh, S.Z., 1992. Geodesic monotone vector fields. Lobachevskii J. Math. 5, 13-28.
[23] Noor, M.A., Zainab, S., Yao, Y., 2012. Implicit methods for equilibrium problems on Hadamard manifolds. J. of Appl. Math. Volume 2012, Article ID 437391, 9 pages.
[24] Noor, M.A., 2005. Generalized mixed quasi-equilibrium problems with trifunction. Appl. Math. Let. 18, 695-700.
[25] Noor, M.A., 2003. Mixed Quasi Variational Inequalities. Appl. Math. Comput. 146, 553-578.
[26] Noor, M.A., 2004. Hemiequilibrium Problems. J. Appl. Math. Stochastic Anal. 17, 235-244.
[27] Noor, M.A., 2005. Some Algorithms for Hemiequilibrium Problems. J. Appl. Math. and Comput. 19, 135-146.
[28] Quiroz, E.A.P., Oliveira, P.R., 2012. Proximal point method for minimizing quasiconvex locally Lipschitz functions on Hadamard manifolds, Nonlinear Anal. 75, 5924-5932.
[29] Rapcsák, T., 1997. Nonconvex Optimization and Its Applications, Smooth nonlinear optimization in $\mathbb{R}^n$. Kluwer Academic Publishers, Dordrecht.
[30] Rapcsák, T., 1991. Geodesic convexity in nonlinear optimization. J. Optim. Theory Appl. 69, 169-183.
[31] Sakai,T., 1996. Riemannian geometry. Translations of Mathematical Monographs, Vol. **149**, American Mathematical Society, Providence.
[32] Tang, G.J., Zhou, L.W., Huang, N.J., 2016. Existence results for a class of hemivariational inequality problems on Hadamard manifolds. Optimization doi: Doi:10.1080/02331934.2016.1147036.
[33] C. Udrişte, 1994. Convex functions and optimization methods on Riemannian manifolds. Math. Appl. 297, Kluwer Academic.
[34] Wang, J.H. López, G., Martín-Márquez, V., Li, C., 2010. Monotone and accretive vector fields on Riemannian manifolds. J. Optim. Theory Appl. 146(3), 691708.
[35] Yao, Y., Noor, M.A., Zainab, S. Liou, Y.C., 2009. Mixed equilibrium problems and optimization problems. J. Math. Anal. Appl. 354, 319-329.
[36] Zhou, L.W., Huang, N.J., 2009. Generalized KKM theorems on Hadamard manifolds with applications. URL: `http://www.paper.edu.cn/index.php/default/releasepaper/content/200906-669`.
[37] Zhou, L.W., Huang, N.J., 2013. Existence of solutions for vector optimization on Hadamard manifolds. J. Optim. Theory Appl. 157, 44-53.